

On asymptotics of unitriangular groups

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The unitriangular group $\mathrm{UT}_n(q)$

Let $\mathrm{UT}_n(q)$ be the group of $n \times n$ upper triangular matrices with entries in $\mathbb{F}_q$ and 1s on the diagonal.

Theorem (Isaacs [4]). *All irreducible complex representations of $\mathrm{UT}_n(q)$ have q -power degrees.*

Set

$$\begin{aligned} N_{n,e}(q) &= \# \text{ irreps. of } \mathrm{UT}_n(q) \text{ of degree } q^e \\ N_n(q) &= \# \text{ irreps. of } \mathrm{UT}_n(q) = \# \text{ conjugacy classes of } \mathrm{UT}_n(q) \end{aligned}$$

Question. What can we say about $N_{n,e}(q)$ or $N_n(q)$?

It is known that the largest degree irrep. of $\mathrm{UT}_n(q)$ is $q^{\mu(n)}$, where

$$\mu(n) = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Conjecture (Higman). $N_n(q)$ is a polynomial in q .

Conjecture (Lehrer). $N_{n,e}(q)$ is a polynomial in q .

Asymptotics of $N_{n,e}(q)$

It turns out that $N_{n,e}(q)$ has nice asymptotics when e is large.

Theorem 1. Let $n \geq 1$ and $0 \leq k \leq \frac{m^2}{4}$, where $m = \lceil \frac{n}{2} \rceil$. Then as $q \rightarrow \infty$,

$$N_{n,\mu(n)-k}(q) \sim a_{m,k} q^{n-m+k},$$

where $a_{m,k}$ is the number of 321-avoiding permutations in $\mathfrak{S}_m$ with k inversions.

The range for k above is such that $a_{m,k} > 0$.

Pattern subgroups

Let $\Phi = \{(i, j) \mid 1 \leq i < j \leq n\}$.

A subset $\mathcal{C} \subseteq \Phi$ is a **closed pattern** if $(i, j), (j, k) \in \mathcal{C}$ implies that $(i, k) \in \mathcal{C}$.

For a closed pattern $\mathcal{C}$, we define its associated **pattern subgroup** to be

$$C = \{x \in \mathrm{UT}_n(q) \mid x_{ij} = 0 \text{ for } (i, j) \in \Phi \setminus \mathcal{C}\},$$

which is in fact a subgroup of $\mathrm{UT}_n(q)$.

Set partitions

Let $\Lambda \vdash [n] = \{1, \dots, n\}$ be a set partition.

We call (i, j) an **arc** of Λ if $i < j$ are consecutive elements in the same part of Λ . Let $\mathrm{Arc}(\Lambda)$ be the set of arcs.

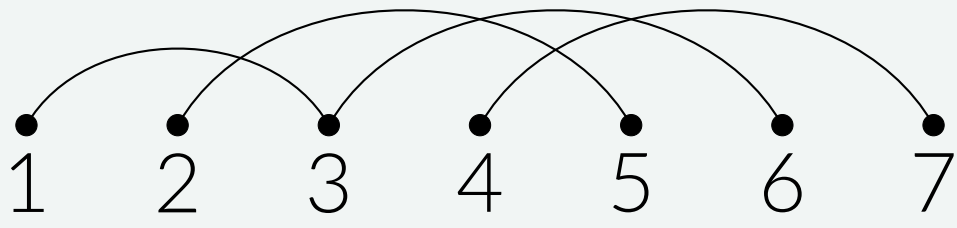
If $(i, k), (j, l) \in \mathrm{Arc}(\Lambda)$, then we say that $((i, k), (j, l))$ is a **crossing** of Λ if $1 \leq i < j < k < l \leq n$. Let

$$\mathrm{Cr}(\Lambda) = \{(i, j) \mid ((i, k), (j, l)) \text{ is a crossing of } \Lambda \text{ for some } k, l\}.$$

An **$\mathbb{F}_q$ -labeled set partition of $[n]$** is a set partition $\Lambda \vdash [n]$ together with a labeling of its arcs by elements of $\mathbb{F}_q^\times$.

Example

For $\Lambda = \{\{1, 3, 6\}, \{2, 5\}, \{4, 7\}\} \vdash [7]$,
 $\mathrm{Arc}(\Lambda) = \{(1, 3), (2, 5), (3, 6), (4, 7)\}$ and $\mathrm{Cr}(\Lambda) = \{(1, 2), (2, 3), (2, 4), (3, 4)\}$.
Its **standard representation** is



Supercharacters partition $\mathrm{Irr}(\mathrm{UT}_n(q))$

There is a supercharacter theory that is useful in studying the irreducible characters of $\mathrm{UT}_n(q)$.

Let $\Lambda \vdash [n]$ be an $\mathbb{F}_q$ -labeled set partition. Then to Λ , there is an associated pattern subgroup N and a linear character $\lambda : N \rightarrow \mathbb{C}^\times$.

The **supercharacter** associated to Λ is the induced character $\xi = \lambda^{\mathrm{UT}_n(q)}$.

The underlying set partition of Λ is the **shape** of ξ .

Theorem (André [1]). *Each irreducible character of $\mathrm{UT}_n(q)$ is a constituent of a unique supercharacter.*

Let $N_{\Lambda,e}(q)$ denote the number of irreducible constituents of degree q^e of any supercharacter of shape Λ – this does not depend on the $\mathbb{F}_q$ -labeling. Then [5]

$$N_{n,e}(q) = \sum_{\Lambda \vdash [n]} (q-1)^{|\mathrm{Arc}(\Lambda)|} N_{\Lambda,e}(q). \quad (\star)$$

The role of crossings

There is an intermediate pattern subgroup $N \triangleleft S \leq \mathrm{UT}_n(q)$, where S is generated by N and the root subgroups corresponding to $\mathrm{Cr}(\Lambda)$.

We know

$$\lambda^S = \sum_{\substack{\phi \in \mathrm{Irr}(S) \\ \langle \phi, \lambda^S \rangle > 0}} \phi(1) \phi.$$

Theorem (André [2]). *Induction defines a bijection between the set of irreducible constituents of λ^S and those of the supercharacter $\xi = \lambda^{\mathrm{UT}_n(q)}$.*

Thus it suffices to understand what happens between N and S .

3-crossings and chains

Λ is **3-noncrossing** if its standard representation does not contain

A **chain** in Λ is a subset of crossings of the form

We have

Theorem 2. *If $\Lambda \vdash [n]$ is a 3-noncrossing set partition, then*

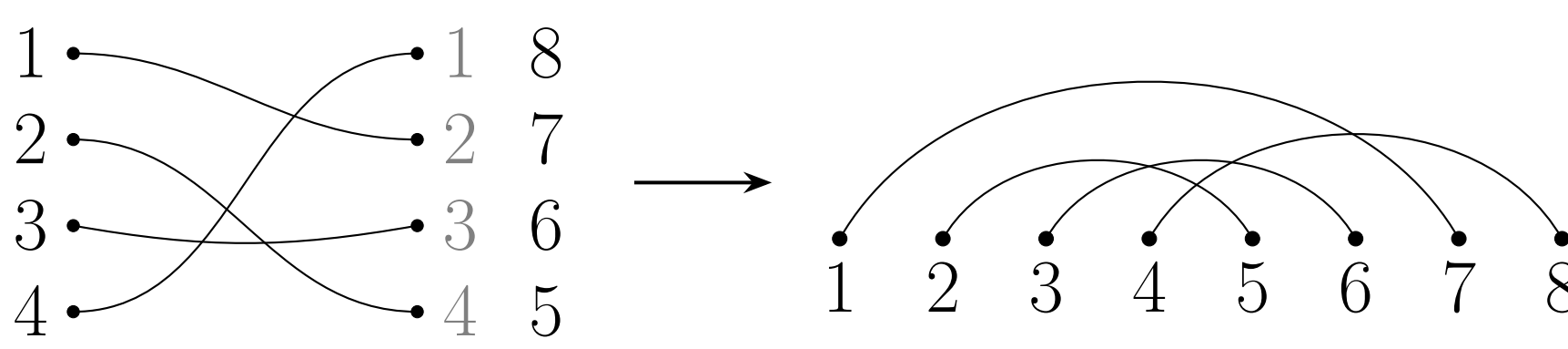
$$N_{\Lambda,e}(q) = \begin{cases} q^{m_{\mathrm{odd}}(\Lambda)} & \text{if } e = d(\Lambda) - |\mathrm{Cr}_{\mathrm{odd}}(\Lambda)|, \\ 0 & \text{otherwise,} \end{cases}$$

where $m_{\mathrm{odd}}(\Lambda)$ denotes the number of maximal chains in Λ of odd length.

Balanced matchings

To show Theorem 1, we need to determine which Λ contribute to the asymptotics in $(\star)$. The even case $n = 2m$ is easier to describe.

Given $\sigma \in \mathfrak{S}_m$, we can obtain $\Lambda \vdash [n]$ by ‘unfolding’ its braid diagram:



We call Λ obtained this way the **balanced matching** associated to σ .

Inversions correspond to crossings, and 321-avoidance corresponds to being 3-noncrossing.

Asymptotics for $N_n(q)$: a conjecture

There is an algorithm [3, 5] to compute $N_{\Lambda,e}(q)$ for small n .

Using this, we noticed that the Λ which contribute to $N_n(q)$ as in $(\star)$ appear to all be 3-noncrossing.

Set $\nu(n) = \lfloor n(n+6)/12 \rfloor$.

Conjecture 1. *For $n \geq 1$, the maximum of $|\mathrm{Arc}(\Lambda)| + m_{\mathrm{odd}}(\Lambda)$ over the collection of 3-noncrossing set partitions $\Lambda \vdash [n]$ is equal to $\nu(n)$.*

Motivated by these observations, Theorem 2, and data from [6],

Conjecture 2. *Let $n \geq 1$. Then as $q \rightarrow \infty$,*

$$N_n(q) \sim K_n q^{\nu(n)},$$

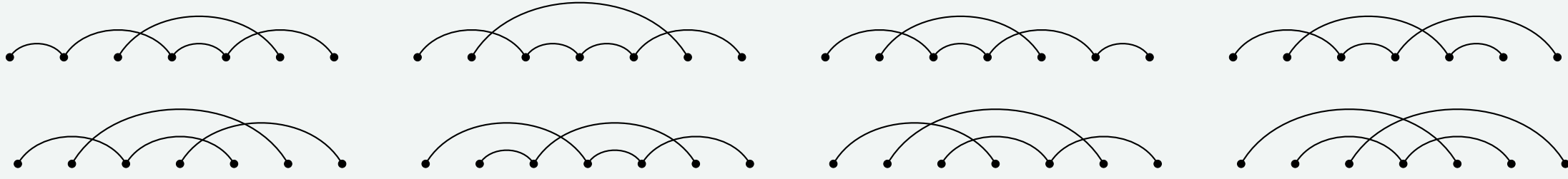
where K_n is the number of 3-noncrossing set partitions $\Lambda \vdash [n]$ with $|\mathrm{Arc}(\Lambda)| + m_{\mathrm{odd}}(\Lambda) = \nu(n)$.

Example

The sequence $(K_n)_{n \geq 1}$ starts as

$$1, 1, 1, 2, 5, 1, 8, 4, 3, 5, 11, 2, 13, 7, 5, 8, \dots$$

For $n = 7$, the corresponding $K_7 = 8$ set partitions of the conjecture are



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