

Asymptotic enumeration of characters of unitriangular groups over $\mathbb{F}_q$

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Basic question

Let $\text{UT}_n(q)$ denote the $n \times n$ upper unitriangular group over $\mathbb{F}_q$. E.g.

$$\text{UT}_4(q) = \begin{pmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{pmatrix}$$

Question

What can we say about irreducible representations of $\text{UT}_n(q)$ (over $\mathbb{C}$)?

What was known

Theorem (Isaacs, 1995)

All irreducible representations of $UT_n(q)$ have q -power degrees.

The degrees are q^e for $0 \leq e \leq \mu(n)$ where

$$\mu(n) = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Define

$$N_{n,e}(q) = \# \text{ irreps. of } UT_n(q) \text{ of degree } q^e$$

$$N_n(q) = \# \text{ irreps. of } UT_n(q) = \# \text{ conjugacy classes of } UT_n(q)$$

Previous results on $N_{n,e}(q)$

$N_{n,e}(q) = \# \text{ irreps. of } \text{UT}_n(q) \text{ of degree } q^e$

- $N_{n,0}(q) = q^{n-1}$.
- Marjoram, 1999: $N_{n,e}(q)$ for $e = 1, 2$.
- Isaacs, 2007: $N_{n,e}(q)$ for $e = \mu(n) - 1, \mu(n)$.
- Evseev, 2011: $N_{n,e}(q)$ for $n \leq 13$ and all e , via an algorithm.
- Marberg, 2011: $N_{n,e}(q)$ for $e \leq 8$ (and n sufficiently large), uses Evseev's algorithm.

Conjecture (Lehrer, 1974)

$N_{n,e}(q)$ is a polynomial in q .

Main result

$$N_{n,e}(q) = \# \text{ irreps. of } \text{UT}_n(q) \text{ of degree } q^e$$

We know asymptotics of $N_{n,e}(q)$ for large e :

Theorem (L)

Let $n \geq 1$ and $0 \leq k \leq \frac{m^2}{4}$, where $m = \lceil \frac{n}{2} \rceil$. Then as $q \rightarrow \infty$,

$$N_{n,\mu(n)-k}(q) \sim a_{m,k} q^{n-m+k},$$

where $a_{m,k}$ is the number of 321-avoiding permutations in $\mathfrak{S}_m$ with exactly k inversions.

321-avoiding permutations

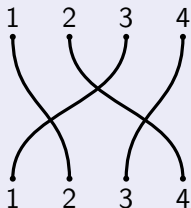
Definition

$\sigma \in \mathfrak{S}_m$ is **321-avoiding** if there are no $1 \leq i < j < k \leq m$ such that $\sigma(i) > \sigma(j) > \sigma(k)$.

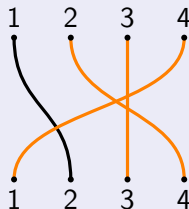
Example

$3142 \in \mathfrak{S}_4$ is 321-avoiding, but $4132 \in \mathfrak{S}_4$ is not.

Braid diagram of 3142



... of 4132



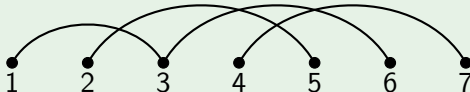
Supercharacter theory for $UT_n(q)$

There is a collection $(\xi_\Lambda)_\Lambda$ of characters of $UT_n(q)$ called *supercharacters*. They are indexed by $\mathbb{F}_q$ -labeled *set partitions* Λ of $[n] = \{1, \dots, n\}$.

Example

$\Lambda = \{\{1, 3, 6\}, \{2, 5\}, \{4, 7\}\}$ is a set partition of $[7]$.

Its *standard representation* is



It has the set of *arcs* $\text{Arc}(\Lambda) = \{(1, 3), (2, 5), (3, 6), (4, 7)\}$, and set of *crossings* $\text{Cr}(\Lambda) = \{(1, 2), (2, 3), (2, 4), (3, 4)\}$.

An $\mathbb{F}_q$ -*labeling* of Λ is a function $\text{Arc}(\Lambda) \rightarrow \mathbb{F}_q^\times$.

Supercharacters partition $\text{Irr}(\text{UT}_n(q))$

Theorem (André, Yan)

For any irreducible character χ of $\text{UT}_n(q)$, there is a unique $\mathbb{F}_q$ -labeled set partition $\Lambda \vdash [n]$ such that χ is a constituent of the supercharacter ξ_Λ .

If two $\mathbb{F}_q$ -labeled set partitions Λ, Λ' have the same underlying set partition, then ξ_Λ and $\xi_{\Lambda'}$ are related by an automorphism, hence decompose in the same way.

For a set partition Λ , let

$$N_{\Lambda,e}(q) = \# \text{ irreducible constituents of } \xi_\Lambda \text{ of degree } q^e.$$

Then

$$N_{n,e}(q) = \sum_{\Lambda \vdash [n]} (q-1)^{|\text{Arc}(\Lambda)|} N_{\Lambda,e}(q).$$

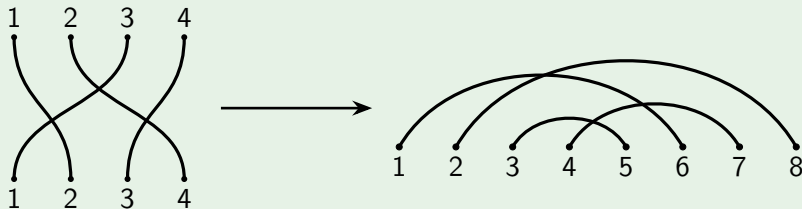
Suffices to understand $N_{\Lambda,e}(q)$.

Obtaining most large degree characters

For simplicity, assume $n = 2m$ is even.

For $\sigma \in \mathfrak{S}_m$, let Λ be the set partition of $[n]$ obtained by 'unfolding' the braid diagram of σ .

Example ($\sigma = 3142$)



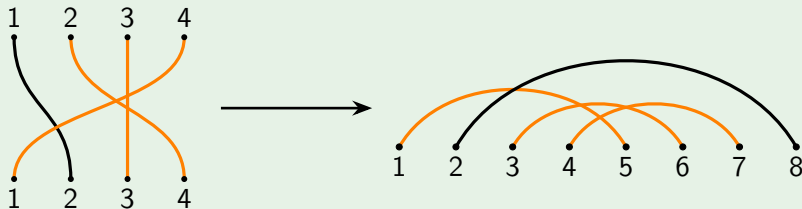
Call Λ obtained this way the *balanced matching* with associated permutation σ .

Obtaining most large degree characters

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Example ($\sigma = 4132$)



Call Λ obtained this way the *balanced matching* with associated permutation σ .

Proof sketch

Theorem (L)

For $0 \leq k \leq \frac{m^2}{4}$, we have $N_{n,\mu(n)-k}(q) \sim a_{m,k} q^{n-m+k}$.

Result follows from

$$N_{n,e}(q) = \sum_{\Lambda \vdash [n]} (q-1)^{|\text{Arc}(\Lambda)|} N_{\Lambda,e}(q),$$

and

Lemma

Let $\Lambda \vdash [n]$ be a set partition and let $k \geq 0$. Then

- if Λ is a balanced matching associated to a 321-avoiding permutation σ with k inversions, then $N_{\Lambda,\mu(n)-k}(q) = q^{n-|\text{Arc}(\Lambda)|+k-m}$;
- otherwise, $N_{\Lambda,\mu(n)-k}(q) \leq q^{n-|\text{Arc}(\Lambda)|+k-m-1}$.

The definition of ξ_Λ

Fix a nontrivial linear character $\psi : \mathbb{F}_q \rightarrow \mathbb{C}^\times$.

Let Λ be an $\mathbb{F}_q$ -labeled set partition of $[n]$.

$\xi_\Lambda = \lambda^{\text{UT}_n(q)}$ is induced from a linear character λ on a subgroup $N = N_\Lambda$.

Example

Let $\Lambda = \{\{1, 3, 6\}, \{2, 5\}, \{4, 7\}\}$ with arc (i, j) labeled by $a_{ij} \in \mathbb{F}_q$. Then

$$N = \begin{pmatrix} 1 & 0 & * & * & * & * & * \\ & 1 & 0 & 0 & * & * & * \\ & & 1 & 0 & 0 & * & * \\ & & & 1 & 0 & 0 & * \\ & & & & 1 & * & * \\ & & & & & 1 & * \\ & & & & & & 1 \end{pmatrix}$$

and $\lambda(x) = \prod_{(i,j) \in \text{Arc}(\Lambda)} \psi(a_{ij}x_{ij})$.

An intermediate subgroup I

There is a subgroup $N \leq S \leq \text{UT}_n(q)$ such that

Theorem (André)

$$\begin{array}{ccc} \{\phi \in \text{Irr}(S) \mid \langle \phi, \lambda^S \rangle > 0\} & \longrightarrow & \{\chi \in \text{Irr}(\text{UT}_n(q)) \mid \langle \chi, \xi_\lambda \rangle > 0\} \\ \phi & \longmapsto & \phi|_{\text{UT}_n(q)} \end{array}$$

is a bijection. Moreover, this bijection preserves multiplicities.

It turns out that

$$\lambda^S = \sum_{\substack{\phi \in \text{Irr}(S) \\ \langle \phi, \lambda^S \rangle > 0}} \phi(1)\phi.$$

An intermediate subgroup II

Example

Let $\Lambda = \{\{1, 3, 6\}, \{2, 5\}, \{4, 7\}\}$. Then

$$S = \begin{pmatrix} 1 & * & * & * & * & * & * \\ & 1 & * & * & * & * & * \\ & & 1 & * & 0 & * & * \\ & & & 1 & 0 & 0 & * \\ & & & & 1 & * & * \\ & & & & & 1 & * \\ & & & & & & 1 \end{pmatrix}$$

3-crossings and chains

Definition

Λ is **3-noncrossing** if it does not contain 

Definition

A **chain** in Λ is a subset of crossings of the form 

Theorem (L)

If Λ is a 3-noncrossing set partition, then

$$N_{\Lambda, e}(q) = \begin{cases} q^{m_{\text{odd}}(\Lambda)} & \text{if } e = d(\Lambda) - |\text{Cr}_{\text{odd}}(\Lambda)|, \\ 0 & \text{otherwise,} \end{cases}$$

where $m_{\text{odd}}(\Lambda)$ denotes the number of maximal chains in Λ of odd length.

Obtaining most characters (conjecturally)

$N_n(q) = \# \text{ irreps. of } \text{UT}_n(q) = \# \text{ conjugacy classes of } \text{UT}_n(q)$

Observation (Vera-López-Arregi, Pak-Soffer)

For $n \leq 16$, $N_n(q)$ is a polynomial in q of degree $\nu(n)$, where

$$\nu(n) = \left\lfloor \frac{n(n+6)}{12} \right\rfloor.$$

Observation

For $n \leq 16$, 'most' irreducible characters are constituents of supercharacters ξ_Λ with Λ 3-noncrossing.

Asymptotics of $N_n(q)$

Conjecture

For $n \geq 1$, the maximum of $|\text{Arc}(\Lambda)| + m_{\text{odd}}(\Lambda)$ over the collection of 3-noncrossing set partitions $\Lambda \vdash [n]$ is equal to $\nu(n)$.

Let

$$K_n = \#\{\Lambda \vdash [n] \mid \Lambda \text{ is 3-noncrossing and } |\text{Arc}(\Lambda)| + m_{\text{odd}}(\Lambda) = \nu(n)\}.$$

Conjecture

Let $n \geq 1$. Then as $q \rightarrow \infty$,

$$N_n(q) \sim K_n q^{\nu(n)}.$$

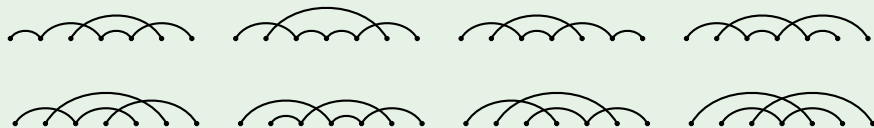
The sequence K_n

The sequence $(K_n)_{n \geq 1}$ starts as

1, 1, 1, 2, 5, 1, 8, 4, 3, 5, 11, 2, 13, 7, 5, 8, $\dots$

Example

For $n = 7$, the corresponding $K_7 = 8$ set partitions of the conjecture are



Thank you!